

# Some Concepts in List Coloring

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## Abstract

In this paper uniquely list colorable graphs are studied. A graph  $G$  is called to be uniquely  $k$ -list colorable if it admits a  $k$ -list assignment from which  $G$  has a unique list coloring. The minimum  $k$  for which  $G$  is not uniquely  $k$ -list colorable is called the  $m$ -number of  $G$ . We show that every triangle-free uniquely colorable graph with chromatic number  $k + 1$ , is uniquely  $k$ -list colorable. A bound for the  $m$ -number of graphs is given, and using this bound it is shown that every planar graph has  $m$ -number at most 4. Also we introduce list criticality in graphs and characterize all 3-list critical graphs. It is conjectured that every  $\chi'_\ell$ -critical graph is  $\chi'$ -critical and the equivalence of this conjecture to the well known list coloring conjecture is shown.

## 1 Introduction

We consider finite, undirected simple graphs. For necessary definitions and notations we refer the reader to standard texts such as [11].

By a  $k$ -list assignment  $L$  to a graph  $G$  we mean a map which assigns to each vertex  $v$  of  $G$  a set  $L(v)$  of size  $k$ . A list coloring for  $G$  from  $L$ , or an  $L$ -coloring for short, is a proper coloring  $c$ , in which for each vertex  $v$ ,  $c(v)$  is chosen from  $L(v)$ . A graph  $G$  is called  $k$ -choosable if it has a list coloring from any  $k$ -list assignment to it. The minimum number  $k$  for which  $G$  is  $k$ -choosable is called the list chromatic number of  $G$  and is denoted by  $\chi_\ell(G)$ . In the following theorem all 2-choosable graphs are characterized.

Before we state the theorem it should be noted that the core of a graph is a subgraph which is obtained by repeatedly deleting a vertex of degree 1, until no vertex of degree 1 remains.

**Theorem A.** [4] *A connected graph is 2-choosable, if and only if its core is either a single vertex, an even cycle, or  $\theta_{2,2,2r}$ , for some  $r \geq 1$ .*

A graph  $G$  is called uniquely  $k$ -list colorable, or  $UkLC$  for short, if it admits a  $k$ -list assignment  $L$  such that  $G$  has a unique  $L$ -coloring. This concept was introduced by Dinitz and Martin [3] and independently by Mahdian and Mahmoodian ([7] and [8]). A characterization of uniquely 2-list colorable graphs follows.

**Theorem B.** [7] *A graph  $G$  is not  $U2LC$  if and only if each of its blocks is either a cycle, a complete graph, or a complete bipartite graph.*

It is easy to see that for each graph  $G$  there exists a number  $k$  such that  $G$  is not  $UkLC$ . The minimum  $k$  with this property is called the  $m$ -number of  $G$  and is denoted by  $m(G)$ . It is shown in [8] that every planar graph has  $m$ -number at most 5, and it is asked about the existence of planar graphs with  $m$ -number equal to 5. We study uniquely list colorable graphs in Section 2, where we prove that every triangle-free uniquely  $(k + 1)$ -colorable graph is uniquely  $k$ -list colorable. We also show that every planar graph has  $m$ -number at most 4, so the answer to that question in [8] is negative.

In Section 3 we introduce list critical graphs and characterize 3-list critical graphs. Finally we pose a conjecture about list critical graphs which is shown to be equivalent to the list coloring conjecture.

## 2 Uniquely list colorable graphs

In [6] one can find several examples of  $UkLC$  graphs, for some arbitrary positive integer  $k$ . In the following lemma we also introduce a class of  $UkLC$  graphs. In this way we relate uniquely list colorable graphs to uniquely colorable graphs.

**Lemma 1.** *Let  $G$  be a uniquely colorable graph with chromatic number  $k+1$ , and  $c$  be its unique  $(k + 1)$ -coloring with color classes  $C_1, \dots, C_{k+1}$ . If for each  $i \leq k + 1$ ,  $|C_i| \geq i - 1$ , then  $G$  is a uniquely  $k$ -list colorable graph.*

**Proof.** We proceed by induction on  $k$  and prove that there exists a  $k$ -list assignment to such graph  $G$  using exactly  $k + 1$  colors, which induces a unique list coloring. For  $k = 1$  the result obviously holds. Let  $G$  be a uniquely  $(k+1)$ -colorable graph as in the statement and  $k \geq 2$ . By induction  $G \setminus C_{k+1}$  admits a  $(k - 1)$ -list assignment  $L'$  which induces a unique list coloring and uses colors  $1, \dots, k$ . For each  $v \in V(G) \setminus C_{k+1}$ , assign the list  $L(v) = L'(v) \cup \{k + 1\}$  to  $v$ , and since  $|C_{k+1}| \geq k$ , it is possible to assign some lists to  $C_{k+1}$  such that  $\bigcap_{v \in C_{k+1}} L(v) = \{k + 1\}$ . Now it is easy to see that  $L$  is the desired list assignment. ■

It is shown in [10] that for every  $k \geq 3$ , in a triangle-free uniquely  $k$ -colorable graph, each color class has at least  $k + 1$  vertices. Using this result, we obtain the following theorem.

**Theorem 1.** *Every triangle-free uniquely  $(k+1)$ -colorable graph is uniquely  $k$ -list colorable.*

On the other hand in [2] it is shown that for each  $k \geq 2$ , there exists a uniquely  $k$ -colorable graph with arbitrary large girth. So by theorem above, for each  $k$ , there exists a  $UkLC$  graph with arbitrary large girth.

We need here a definition which is a generalization of the concept of a  $UkLC$  graph.

**Definition 1.** *Let  $G$  be a graph and  $f$  a function from  $V(G)$  to  $\mathbb{N}$ . An  $f$ -list assignment  $L$  to  $G$  is a list assignment in which  $|L(v)| = f(v)$  for each vertex  $v$ . The graph  $G$  is called to be uniquely  $f$ -list colorable, or  $UfLC$  for short, if there exists an  $f$ -list assignment  $L$  for it such that  $G$  has a unique  $L$ -coloring.*

By definition above, if  $G$  is a  $UfLC$  graph, where  $f(v) = k$  for each vertex  $v$  of  $G$ , then  $G$  in fact is a  $UkLC$  graph. To prove the next theorem, we need a relation which is proved in Truszczyński [9] and states that if  $G$  is a uniquely  $k$ -colorable graph, then  $e(G) \geq (k - 1)n(G) - \binom{k}{2}$ .

**Theorem 2.** *If  $G$  is a  $UfLC$  graph, then*

$$\sum_{v \in V(G)} f(v) \leq n(G) + e(G).$$

**Proof.** Suppose that  $L$  is an  $f$ -list assignment to  $G$  using colors  $1, 2, \dots, t$ , such that  $G$  has a unique  $L$ -coloring. We construct a uniquely  $t$ -colorable graph  $G^*$  as follows. Let  $V(G) = \{v_1, \dots, v_n\}$  and  $K_t$  is a complete graph on the vertex set  $\{w_1, \dots, w_t\}$ . Now for  $G^*$  consider the union of  $G$  and  $K_t$  and add edges  $v_i w_j$  where  $1 \leq i \leq n$ ,  $1 \leq j \leq t$ , and  $j \notin L(v_i)$ .

Consider a  $t$ -coloring  $c$  of  $G^*$ . Without loss of generality we can assume that  $c(w_i) = i$  for each  $1 \leq i \leq t$ . Since  $G$  has a unique  $L$ -coloring, by construction of  $G^*$ ,  $c$  is the only  $t$ -coloring of  $G^*$ . So  $G^*$  is a uniquely  $t$ -colorable graph. On the other hand  $G^*$  has  $n(G) + t$  vertices, and  $e(G) + \binom{t}{2} + \sum_{v \in V(G)} (t - f(v))$  edges. Therefore as mentioned above, we have

$$e(G) + \binom{t}{2} + \sum_{v \in V(G)} (t - f(v)) \geq (n(G) + t)(t - 1) - \binom{t}{2}$$

and after simplification we obtain the result.  $\blacksquare$

A natural question which arises here is that whether or not equality holds in Theorem 2? In the following proposition we give a positive answer to this question.

**Proposition 1.** *For every graph  $G$ , there exists  $f : V(G) \rightarrow \mathbb{N}$  such that  $G$  is  $UfLC$  and  $\sum_{v \in V(G)} f(v) = n(G) + e(G)$ .*

**Proof.** We proceed by induction on the number of vertices of  $G$ . For  $n(G) = 1$  the statement is obvious. Consider a graph  $G$  with  $n(G) \geq 2$  and a vertex  $v$  of  $G$ . By induction there exists  $f' : V(G \setminus v) \rightarrow \mathbb{N}$  and an  $f'$ -list assignment  $L'$  to  $G \setminus v$  such that  $G \setminus v$  has a unique  $L'$ -coloring, and  $\sum_{w \in V(G \setminus v)} f'(w) = n(G \setminus v) + e(G \setminus v)$ . Consider a color  $a$  which is not used by  $L'$ , and define a list assignment  $L$  to  $G$  as follows.

$$L(w) = \begin{cases} a & \text{for } w = v \\ L'(w) \cup \{a\} & \text{for } w \in N(v) \\ L'(w) & \text{otherwise.} \end{cases}$$

It is easy to verify that  $G$  has a unique  $L$ -coloring and that we have  $\sum_{v \in V(G)} |L(v)| = n(G) + e(G)$ .  $\blacksquare$

Although the proposition above shows that in Theorem 2 equality may hold, but it seems that if  $f(v) = k$  for each vertex  $v$ , equality does not hold and we have  $e(G) > (k - 1)n(G)$ .

By definition every graph  $G$  for  $k = m(G) - 1$  is UkLC. So by Theorem 2 we have the following.

**Theorem 3.** *For a graph  $G$  let  $\bar{d}(G)$  denote the average degree of  $G$ , i.e.  $\bar{d}(G) = 2e(G)/n(G)$ . Then*

$$m(G) \leq \lfloor \frac{\bar{d}(G)}{2} \rfloor + 2.$$

For example suppose that  $G$  is a bipartite graph. We have  $\bar{d}(G) \leq n(G)/2$  so Theorem 3 implies  $m(G) \leq \lfloor n(G)/4 + 2 \rfloor$ . This bound can be improved to a logarithmic bound as we will show in Theorem 4, but first we need a lemma.

Let  $L$  be a  $k$ -list assignment to a graph  $G$  such that  $G$  has a unique  $L$ -coloring  $c$ . For each vertex  $v$  of  $G$ , all the elements of  $L(v) \setminus \{c(v)\}$  must appear in  $N(v)$ , so if we denote by  $c_N(v)$  the set of colors appearing in  $N(v)$ , then  $|c_N(v)| \geq k - 1$ . In the following lemma we state a stronger result.

**Lemma 2.** *Suppose that  $G$  is a UkLC graph, and  $L$  is a  $k$ -list assignment to  $G$  such that  $G$  has a unique  $L$ -coloring  $c$  with color classes  $C_1, \dots, C_t$  such that  $c(C_i) = \{i\}$ . There exist at least  $k - 1$  classes containing a vertex  $v$  with  $|c_N(v)| \geq k$ .*

**Proof.** Without loss of generality suppose that for  $\ell \geq k - 1$ ,  $C_\ell$  contains no vertex  $v$  with  $|c_N(v)| \geq k$ . Assume that  $u \in C_{k-1}$ ,  $i = c(v_0)$ , and  $j \in L(v_0) \setminus \{1, \dots, k-1\}$ . Suppose that  $G_{ij}$  is the subgraph of  $G$  induced on  $C_i \cup C_j$ . Since for each vertex  $v$  of the component of  $G_{ij}$  containing  $v_0$  we have  $|c_N(v)| = k - 1$ , it is implied that  $i, j \in L(v)$ . So we can interchange the colors  $i$  and  $j$  in this component to obtain a new  $L$ -coloring for  $G$ . This contradiction completes the proof.  $\blacksquare$

It is shown in [4] that every non- $k$ -choosable bipartite graph has more than  $2^{k-1}$  vertices. So by applying Lemma 2, we deduce the following theorem.

**Theorem 4.** *Let  $G$  be a bipartite graph. Then  $m(G) \leq 2 + \log_2 n(G)$ .*

**Proof.** Suppose that  $L$  is a  $k$ -list assignment to  $G$  such that  $G$  has a unique  $L$ -coloring  $c$ . By Lemma 2,  $G$  has a vertex  $v_0$ , such that there are

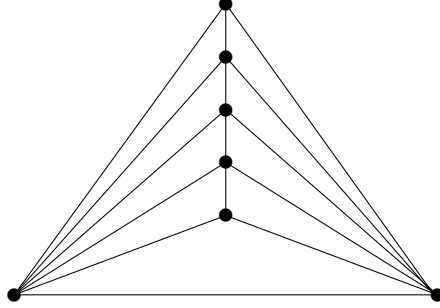


Figure 1: A uniquely 3-list colorable planar graph

at least  $k$  colors appeared at  $N(v_0)$  in  $c$ . Let  $G'$  be the graph obtained from  $G$ , by duplicating  $v_0$ , i.e. adding a new vertex  $w$  to  $G$  and joining it to  $N(v_0)$ . Now assign to  $w$  a list containing  $k$  of the colors appeared at  $N(v_0)$  in  $c$ , and the list  $L(v)$  to each other vertex  $v$  of  $G'$ . It is clear that  $G'$  is a bipartite graph and it has no coloring from these lists, so it is not  $k$ -choosable. Hence  $n(G') > 2^{k-1}$  vertices. This implies that  $n(G) \geq 2^{k-1}$ , and so  $k \leq 1 + \log_2 n(G)$ . Now we obtain the desired relation by setting  $k = m(G) - 1$ . ■

In the remainder of this section we state some consequences of Theorems 2 and 3.

It is well known that a planar graph with  $n$  vertices has at most  $3n - 6$  edges. So the following theorem is an immediate consequence of Theorem 3.

**Theorem 5.** *For every planar graph  $G$  we have  $m(G) \leq 4$ .*

By Lemma 1 the planar graph shown in Figure 1 is a U3LC graph for which the inequalities in Theorem 3 and Theorem 5 turn to be equalities.

Furthermore we know that a triangle-free planar graph  $G$ , has at most  $2n(G) - 4$  edges. So Theorem 3 implies that each triangle-free planar graph  $G$  has  $m$ -number at most 3. In the following proposition a stronger result is obtained.

**Proposition 2.** *If a plane graph has at most 7 triangular faces, then  $m(G) \leq 3$ .*

**Proof.** Consider a U3LC plane graph  $G$  with  $n$  vertices,  $e$  edges,  $f$  faces, and  $t$  triangular faces. We have  $2e \geq 4(f - t) + 3t = 4f - t$ , and by Euler

formula  $f = 2 - n + e$ , so  $t \geq 8 - 4n + 2e$ . On the other hand Theorem 2 implies that  $e \geq 2n$ . So  $t \geq 8$ , as desired.  $\blacksquare$

The following conjecture is about the structure of U3LC planar graphs which is motivated by the proposition above.

**Conjecture 1.** *Every U3LC planar graph has  $K_4$  as a subgraph.*

For another application of Theorem 2, we study line and total versions of uniquely list coloring.

A graph  $G$  is called to be uniquely  $k$ -list edge colorable, if  $L(G)$  is a uniquely  $k$ -list colorable graph. The edge  $m$ -number of  $G$  is defined to be  $m(L(G))$ , and is denoted by  $m'(G)$ . It is straightforward to see that for each graph  $G$ ,  $\bar{d}(L(G)) \leq \Delta(L(G)) \leq 2\Delta(G) - 2$ . So using Theorem 3 we deduce the following.

**Theorem 6.** *For every graph  $G$ , we have  $m'(G) \leq \Delta(G) + 1$  and if  $m'(G) = \Delta(G) + 1$  then  $G$  is a regular graph.*

Note that in Theorem 6 it is shown that if  $G$  is not a regular graph, then  $m(G) \leq \Delta(G)$ . So in this case  $m(G) \leq \chi(G)$ .

### 3 List critical graphs

In this section we introduce a concept of list critical graphs and we state some results concerning it.

**Definition 2.** *A graph  $G$  is called  $\chi_\ell$ -critical if for each proper subgraph  $H$  of it we have  $\chi_\ell(H) < \chi_\ell(G)$ .*

We sometimes refer to a  $\chi_\ell$ -critical graph  $G$  as a  $k$ -list critical graph, where  $k = \chi_\ell(G)$ . It can easily be verified that the only connected 2-list critical graph is  $K_2$ , odd cycles are 3-list critical, and the complete graph  $K_k$  is  $k$ -list critical.

Obviously every graph  $G$  contains a  $\chi_\ell$ -critical subgraph  $H$  such that  $\chi_\ell(H) = \chi_\ell(G)$ , and by an argument similar to critical graphs,  $\delta(G) \geq \chi_\ell(G) - 1$ . On the other hand there exists some differences between critical graphs and list critical graphs. For example it is well known that every

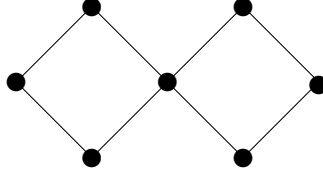


Figure 2: A non-2-connected 3-list critical graph

critical graph is 2-connected. In Figure 2 we have given an example of a 3-list critical graph which is not 2-connected.

In the next theorem 3-list critical graphs are characterized.

**Theorem 7.** *A graph is 3-list critical if and only if it is either an odd cycle, two even cycles with a path joined them,  $\theta_{r,s,t}$  where  $r, s, t$  have the same parity, and at most one of them is 2, or  $\theta_{2,2,2,2r}$  where  $r \geq 1$ .*

**Proof.** By use of Theorem A, it is easy to see that all the graphs listed in the statement are 3-list critical.

For the converse suppose that  $G$  is a 3-list critical graph. If  $G$  is 2-connected, by a theorem of Whitney [12]  $G$  has an ear decomposition  $K_2 \cup P^1 \cup \dots \cup P^q$ . If  $q \geq 4$ , deleting an edge of  $P^q$ , yields a non-2-choosable graph, which contradicts the 3-list criticality of  $G$ . So  $q \leq 3$  and we consider the following three cases.

- If  $q = 1$ ,  $G$  is a cycle, and so it is an odd cycle.
- If  $q = 2$ ,  $G = \theta_{r,s,t}$ . In this case by deleting each edge of  $G$ , we obtain a graph whose core is a cycle, and since this cycle must be even, the numbers  $r, s$ , and  $t$  have the same parity. Now if at least two of  $r, s$ , and  $t$  are equal to 2, we have  $\chi_\ell(G) = 2$ , a contradiction.
- The last case is  $q = 3$ . By deleting each edge of  $G$  we obtain a graph whose core is a  $\theta_{r,s,t}$ , and since this graph must be 2-choosable, we have  $r = s = 2$  and  $t$  is an even number. Now by case analysis, it is easy to see that  $G = \theta_{2,2,2,2\ell}$ .

On the other hand if  $G$  is not 2-connected, we consider two end-blocks  $B_1$  and  $B_2$  of  $G$ . Since  $\delta(G) \geq 2$  each of  $B_1$  and  $B_2$  has a cycle. So  $G$  has a subgraph  $H$  which is composed of two edge-disjoint cycles joined to each



other by a path (possibly of length zero). We know that  $\chi_\ell(H) = 3$ , and so by  $\chi_\ell$ -criticality of  $G$ ,  $G$  has no edge outside  $H$ , i.e.  $G = H$ . Hence  $G$  satisfies the statement.  $\blacksquare$

Suppose that  $G$  is a  $k$ -list critical graph, and  $L$  is a  $k$ -list assignment to  $G$ . Consider a vertex  $v$  in  $G$  and a color  $a \in L(v)$ . Assign to each vertex  $u$  in  $G \setminus v$  the list  $L(u) \setminus \{a\}$ . Since  $G \setminus v$  is  $(k-1)$ -choosable, it has a coloring from the assigned lists, and one can extend this coloring to an  $L$ -coloring of  $G$  by assigning the color  $a$  to  $v$ . So there exists an  $L$ -coloring for  $G$  in which  $v$  takes  $a$ .

As mentioned in the previous paragraph, every  $k$ -list critical graph has at least  $k$  colorings from each  $k$ -list assignment so every  $k$ -list critical graph has m-number at most  $k$ .

A graph  $G$  is called to be **edge  $k$ -choosable**, if the graph  $L(G)$  is  $k$ -choosable, and the **list chromatic index** of  $G$  written  $\chi'_\ell(G)$  is defined to be  $\chi_\ell(L(G))$ . As in the case of defining  $\chi'$ -critical graphs, one can define a  $\chi'_\ell$ -critical graph  $G$  to be a graph in which for each proper subgraph  $H$ ,  $\chi'_\ell(H) < \chi'_\ell(G)$ . We recall here the well known List Coloring Conjecture (LCC), which first appeared in print in [1].

**Conjecture.** [1] *Every graph  $G$  satisfies  $\chi'_\ell(G) = \chi'(G)$ .*

Suppose that  $G$  is a counterexample to the LCC with minimum number of edges. So for each edge  $uv$  of  $G$  we have  $\chi'_\ell(G \setminus uv) = \chi'(G \setminus uv)$ , and since  $\chi'(G \setminus uv) \leq \chi'(G) < \chi'_\ell(G)$ , we conclude that  $\chi'_\ell(G \setminus uv) = \chi'_\ell(G) - 1$ . This means that  $G$  is a  $\chi'_\ell$ -critical graph and therefore  $\chi'_\ell$ -critical graphs may be useful to attack the LCC.

In the study of  $\chi'_\ell$ -critical graphs we have lead to the following conjecture.

**Conjecture 2.** *Every  $\chi'_\ell$ -critical graph is  $\chi'$ -critical.*

**Proposition 3.** *The conjecture above is equivalent with the LCC, while its converse is implied by the LCC.*

**Proof.** It is straight forward to check that the list coloring conjecture implies Conjecture 2 and its converse. On the other hand suppose that

Conjecture 2 is true, and  $G$  is a counterexample to the list coloring conjecture with minimum number of edges. As mentioned above  $G$  is  $\chi'_\ell$ -critical, and by Conjecture 2, it is  $\chi'$ -critical. By removing an arbitrary edge  $uv$  from  $G$  we obtain a graph for which the list coloring conjecture holds. So  $\chi'_\ell(G \setminus uv) = \chi'(G \setminus uv)$ , and this means that  $\chi'_\ell(G) - 1 = \chi'(G) - 1$ , a contradiction. ■

In [5] it is proved that every bipartite multigraph fulfills the LCC. On the other hand we know that the only bipartite  $\chi'$ -critical graphs are stars. So the following theorem is implied by a similar argument as in the previous paragraph, the only bipartite  $\chi'_\ell$ -critical graphs are stars.

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